# Boundedness of moduli of Varieties of general type

- Preliminaries
  - Volume
  - Deformation invariance
  - DCC sets
- semi log canonical varieties.

Numerical dimension. $D$ ps-eff. $\dim X = n$.

$$\kappa_\sigma(X,D) := \max_{H \in \mathrm{Pic}(X)} \left\{ k \in \mathbb{N} \,\middle|\, \limsup \frac{h^0(X, \mathcal{O}_X(mD + H))}{m^k} > 0 \right\}$$

$D$ nef.

$$\kappa_\sigma(X,D) = \nu(X,D) = \max\{ k \in \mathbb{N} \mid H^{n-k} \cdot D^k > 0 \}.$$

for $H$ ample

If $\kappa_\sigma(X,D) = \kappa(X,D)$, $D$ : abundant

Volume $\dim X = n$,

$$\mathrm{vol}(X,D) := \limsup \frac{n!\, h^0(X, \mathcal{O}_X(mD))}{m^n}$$

restricted volume, $V \subset X$ $\dim V = d$

$$\mathrm{Vol}(X|V, D) := \limsup \frac{d!\, \dim\left(\mathrm{Im}\left(H^0(X, \mathcal{O}_X(mD)) \to H^0(\mathcal{O}_V(mD|_V))\right)\right)}{m^d}$$

**Lemma 2.2.7** $f: X \to Z$ bir.

$(X, \Delta)$ $(Z, B)$ lc.

- $K_X + \Delta$ is big.
- $(X, \Delta)$ lc. model $\quad g: X \dashrightarrow Y$.

If $f_* \Delta \leq B$ & $\mathrm{vol}(X, K_X + \Delta) = \mathrm{vol}(K_Z + B)$.

Then:

$$\underbrace{Z \dashrightarrow X \dashrightarrow Y}_{\text{lc model of } (Z, B).}$$

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**Proof** $B_X$ strict trans of $B$ on $X$.

$F$ the $f$-exceptional divisors.

$\pi: W \to X$ log res of $(X, B_X + F)$ and of $g$.

$$K_W + \Theta = \pi^*(K_X + \Delta) + E$$

$(W, \Theta)$. same l.c model as $(X, \Delta)$ and same Volume

replace $(X, \Delta)$ by log smooth pair $g: X \to Y$.

$\underbrace{K_X + B_X + F}_{D}$, $\underline{K_X + D} = f^*(K_Z + B)$.

replace $(Z, B)$ by $(X, D)$.

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$\mathrm{vol}(K_X + \Delta) = \mathrm{vol}(K_X + D)$.

$D - \Delta \geq 0$

$\|$

$\ldots$

$A := g_*(K_X + \Delta)$ <u>ample</u>.

$H := g^*(A)$, $K_X + \Delta - H \geq 0$.

<u>Claim</u>

$D - \Delta$ is $g$-<u>exceptional</u>!

$S$ a component in $L$ w/ coeff $a > 0$

$v(t) = \mathrm{vol}(X, H + tS)$.

$$\underline{V(0)} = \mathrm{vol}(X, H)$$
$$= \mathrm{Vol}(X, K_X + \Delta)$$
$$= \mathrm{vol}(K_X + D)$$
$$\geq \mathrm{Vol}((H + L))$$
$$\geq \mathrm{Vol}(H + aS)$$
$$= \underline{V(a)}.$$

$K_X + D \geq H + L$

$V$ is constant near $0$.

$$\underline{0} : \frac{1}{n}\frac{dv}{dt}\Big|_{t=0} = \mathrm{Vol}_{X|\underline{S}}(H) \geq S \cdot \underline{H}^{n-1}$$
$$= g_* S \cdot \boxed{A^{n-1}}$$

$g_* S = 0 \Rightarrow L$ is $g$-exceptional.

Deformation invariance:

**Lemma 2.3.1** $\pi : X \to U$.

$(X, \Delta)$ log smooth over $U$.

$A$ rel. ample. $(X, \Delta + A')$ is log sm. over $U$.

$A' \sim \lfloor \Delta \rfloor + A$.

Then:

$f_*(\mathcal{O}_X(m(K_X+\Delta)+A)$

$\twoheadrightarrow H^0(X_u, \mathcal{O}_{X_u}(m(K_{X_u}+\Delta_u)+A_u)$

$\forall u \in U$.

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Proof.

$$m(K_X+\Delta)+A \sim m\Big(K_X+\underbrace{\Delta - \tfrac{1}{m}\lfloor \Delta \rfloor + \tfrac{1}{m}A'}_{=:\,\Delta'}\Big)$$

$\Delta \leq \Delta' \leq \Delta + A'$. $(X, \Delta')$ is log smooth

coeff $\Delta' < 1$. $\Delta'$ is big over.

$\Delta' \sim \Delta + \frac{1}{m} \underbrace{A}_{\text{ample.}}$

HMX. (1.8.1).

$h^0(X_u, \mathcal{O}_{X_u}(m(K_{X_u} + \Delta'_u))$

are independent of $u$.

so the rest. is surj. for all $u$.

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Def. $\pi : X \to U$ $D$ div. $C$ prime divisor

For $D$ big.

$\sigma_C(X/U, D) :=$

$$\inf \{ \operatorname{mult}_C(D') \mid D' \sim_{\mathbb{R}, U} D, \ D' \geq 0 \}$$

$D$ is pseudo-effective.

$$\sigma_C(X/U, D) := \lim_{\varepsilon \to 0} \sigma_C(X/U, D + \varepsilon A)$$

$$N_\sigma(X/U, D) := \sum_C \sigma_C(X/U, D)\, C$$

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**Lemma 2.3.2**

$\pi : X \to U$. $(X, \Delta)$ log smooth.

$K_X + \Delta$ ps-eff.

$$N_\sigma(X/U, K_X + \Delta)\big|_{X_u} = N_\sigma(X_u, K_{X_u} + \Delta_u)$$

$u \in U$.

**Proof** $(N_\sigma \quad )\big|_{X_u} \geq \boxed{N_\sigma(X_u, K_{X_u} + \Delta_u)}$

By definition.

Pick $A$, $\lfloor \Delta \rfloor + A \sim A'$

with $(X, \Delta + A')$ log smooth.

$$f_*(\mathcal{O}_X(m(K_X+\Delta)+A) \to H^0(X_u, \ldots$$

$$\Rightarrow (N_\sigma)\big|_{X_u} \leq (N_\sigma(X_u)).$$

$\square$

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**Lemma 2.3.3** $\pi : X \to U$.

$(X, \Delta)$ log smooth over $U_0$

strata of $\Delta$ have irreducible fibers.

$K_X+\Delta$ ps-eff. $\quad 0 \in U$ closed.

$$\Theta_0 := \Delta_0 - \underline{\underline{\Delta_0}} \wedge \underline{N_\sigma(\underline{K_0, K_{X_0}+\Delta_0})}$$

$$0 \leq \Theta \leq \Delta \text{ s.t. } \Theta_0 = \Theta|_{X_0}$$

Then: $\underline{\underline{\Theta}} = \underline{(\Delta - \Delta \wedge N_\sigma(X/U, K_X+\Delta)_{\eta})}$

Proof: Pick ample $H$.

$0 \leq S \leq \lfloor \Delta \rfloor$. exists

$H' \sim H + S$.

s.t $(X, \Delta + H')$ is log smooth

Fix $m \in \mathbb{N}$.

$$\phi_0 := \Delta_0 - \Delta_0 \wedge N_\sigma\left(X_0, K_{X_0} + \Delta_0 + \tfrac{1}{m} H_0\right)$$

$$\boxed{\phi_0 = \phi|_{X_0}} \qquad \boxed{0 \leq \phi \leq \Delta}$$

$$\begin{tikzcd}
\pi_* \mathcal{O}_X(m(K_X + \phi) + H) \arrow[r, hook, "\text{inj.}"] \arrow[d, "\text{surj.}"'] & \pi_*(m(K_X + \Delta) + H) \arrow[d, "\text{surj.}"] \\
H^0(\mathcal{O}_X(m(K_{X_0} + \phi_0) + H_0)) \arrow[r, "\text{surj.}", "\text{bij.}"'] & H^0(m(K_{X_0} + \Delta_0) + H_0)
\end{tikzcd}$$

$\phi \leq \Delta$

We can locally lift the surjection surjection.

$$\pi_*(\mathcal{O}_X(m(K_X+\phi)+H) \twoheadrightarrow)$$

$$\pi_*(\mathcal{O}_X(\underbrace{m(K_X+\Delta)+\tfrac{H}{m}}))$$

$$m(K_X+\phi)+H \geq m(K_X+\Delta)+H - N_\sigma(m(K_X+\Delta))+\tfrac{H}{m}$$

$$\left[\ \phi \geq \Delta - \underbrace{N_\sigma(K_X+\Delta+\boxed{\tfrac{1}{m}H})}\right.$$

$$\phi \geq \Delta - \Delta \wedge$$

$$\theta \geq \Delta - \Delta \wedge N_\sigma(K_X+\Delta).$$

$$\underbrace{(A \wedge B)}|_{x_0} \leq \underbrace{A|_{x_0}} \wedge \underbrace{B|_{x_0}}$$

$$(\Delta - \Delta \wedge N_6)|_{x_0}$$

$$= \Delta_0 - (\Delta \wedge N_6)|_{x_0}$$

$$\geq \Delta_0 - \Delta_0 \wedge N_6(K_{x_0} + \Delta_0)$$

$$\Rightarrow \quad \Theta_0 \leq (\Delta - \Delta \wedge N_6)|_{x_0}$$

by uniqueness

$$\boxed{\Rightarrow} \; \Theta \leq \Delta - \Delta \wedge N_6 .$$

$\square$

**Lemma 2.3.4** $\pi : X \to U$.

$(X, D)$ logsmooth

$\mathrm{coeff}(D) = 1$. $\quad 0 \in U$ closed point.

$$\pi_* \mathcal{O}_X(K_X + D) \to H^0(X_0, \mathcal{O}_X(K_{X_0} + D_0)).$$

Proof: We can cut by hyperplanes

assume $U$ is a curve.

We want:

$$H^0(X, \mathcal{O}_X(K_X + X_0 + D)) \to H^0(X_0, \mathcal{O}_X(K_{X_0} + D_0))$$

is surj.

$$0 \to \mathcal{O}_X(-X_0) \to \mathcal{O}_X \to \mathcal{O}_{X_0} \to 0.$$

$$\otimes (K_X + X_0 + D)$$

$$\to \boxed{H^0(\mathcal{O}_X(K_X+X_0+D)) \xrightarrow{\text{surj.}} H^0(\mathcal{O}_{X_0}(K_{X_0}+D_0))}$$

$$\boxed{\partial} \to H^1(\mathcal{O}_X(K_X+D)) \xrightarrow{\text{inj}} H^1(\mathcal{O}_X(K_X+X_0+D))$$

We want injectivity.

**Kollár's injectivity Lemma.**

$$H^1(X, \mathcal{O}_X(K_X+B+\boxed{mH})) \to H^1(\mathcal{O}_X(K_X+B+(m'+m)H))$$

$m>0.$

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**DCC sets**

**Lemma 2.4.1** $I \subseteq \mathbb{R}$ DCC.

Fix $d$.

$$T_d := \left\{ \boxed{\{d_1, \dots d_k\}} \;\middle|\; k \in \mathbb{N},\ d_i \in I,\ \sum = d \right\}.$$

$T_d$ is finite.

**Proof** $k < N$. constant number

of elements, we take non-dec. sequences of K-tuples.

subsequence has to be constant.

$\Rightarrow$ Finite.

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**Lemma 2.4.2**

$J \subseteq \mathbb{R}_{\leq 1}$ finite.

$$I_\circ := \left\{ a \in [0,1] \;\middle|\; a = 1 + \sum_{i=1}^{K} a_i - K \right\}$$

$a_i \in J$.

is finite.

**Proof** $a_i \neq 1$. $a_i \leq 1 - \varepsilon$.

$\Rightarrow a = 1 + \sum a_i - K \geq 0$

$1 + \sum (a_i - 1) \geq 0$.

$$K \leq \frac{1}{\varepsilon}.$$

$\Rightarrow$ finite possibilities.

# Semi log canonical varieties:

$X$ demi-normal.

- $S_2$.
- in codimension 1, singularities are double normal crossings.

$X$ demi-normal $\Delta$

$K_X + \Delta$.

$n : Y \to X$ normalization.

$$K_Y + \underline{\underline{\Gamma}} = n^*(K_X + \underline{\Delta})$$

$(X, \Delta)$ is slc. if $(Y, \Gamma)$ is $\underline{\underline{\text{l.c.}}}$

## Theorem 2.5.1

$(X, \Delta)$ slc. $n : Y \to X$,

$$K_Y + \Gamma = n^*(K_X + \Delta).$$

$(Y, \Gamma)$ l.c.

$K_X + \Delta$ is semi-ample iff $K_Y + \Gamma$ is semi-ample.

**Proof** $K_X + \Delta$ is semi-ample, then so is $K_Y + \Gamma$.

Sketch: $K_Y + \Gamma$ semi-ample.

$m(K_Y+\Gamma) =: M$ is b.p.f.

$g : Y \to Z$

$g^*(H) = M$ $\quad$ $H$ very ample on $Z$.

double locus on $Y$ has an involution on $\Gamma$.

$$\begin{tikzcd}
\Gamma \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & Y \arrow[r, "q" above, "n_i" below] \arrow[d] & X \arrow[d, "g"] \\
T \arrow[r, shift left] \arrow[r, shift right] & Z \arrow[r, "q"] & W
\end{tikzcd}$$

Kollár's glueing theory.

$Z_H$, $T_H$ to be total spaces of $H$ and $H|_T$.

$$\begin{tikzcd}
T_H \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & Z_H \arrow[r, "\text{quotient}"] \arrow[d] & A \arrow[d] \\
T \arrow[r, shift left] \arrow[r, shift right] & Z \arrow[r, "f"] & W
\end{tikzcd}$$

$A$ line bundle over $W$.

$$f^*(A) = H$$

$$g^*(A) = \mathcal{O}_X(m(K_X + \Delta))$$

$K_X + \Delta$ is semi-ample.

$\square$

Fin!